Unitarization in Nonlinear Spinor Field Quantum Theory

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Heisenberg performed a dipole ghost regularization of the Lee-model and a subsequent unitarization of the corresponding state space. If this method is transferred to the relativistic nonlinear spinor field model, insoluble difficulties appear as long as no explicit state representations are used. In preceding papers such state representations for the spinor field have been derived and in the present paper it is demonstrated how they can be successfully used to perform the relativistic analogon to Heisenberg's Lee-model unitarization. As the state representations have to be derived by construction, the method is demonstrated for the two particle fermion sector which is sufficiently well-known.

To regularize a non-renormalizable nonlinear spinor field, Heisenberg proposed to use a state space with indefinite metric [1]. Among the various concepts to regularize divergent quantum field theories by indefinite metric he used a special one, namely the dipole ghost regularization. Heisenberg first introduced a dipole ghost in the Lee-model and demonstrated that by this procedure regularization is achieved without having unsoluble difficulties with unitarization [2]. Guided by these results, Heisenberg proposed an application of this method to the nonlinear spinor field. But the extension from the Lee-model to nonlinear spinor field theory is not straight-forward. While in the Lee-model a Fock space representation is applied, nonlinear spinor theory has to be evaluated in the frame of general relativistic quantum field theory. Combining the methods of conventional quantum field theory with dipole ghost regularization, it can be shown that serious drawbacks result:

i) Because of non-canonical quantization the derivation of the prepared Green functions hierarchy breaks down [3].

ii) The derivation of the S-matrix from the Green functions is not possible as in LSZ formalism [4].

iii) For unitarization explicit state representations are required.

For the removal of these serious drawbacks the present axiomatic quantum field theory gives no hint. Also, Heisenberg and his co-workers made no proposals to overcome these difficulties. Obviously, a new approach has to be made in order to obtain meaningful results comparable with experiment. This new approach is given by functional quantum theory of the nonlinear spinor field developed by Stumpf and co-workers. In brevity, it consists of the direct construction of the relativistic state space of the corresponding system and it explicitly solves the problems i)—iii). In preceding papers the construction of bound and scattering states [5], of scalar products [6] and of global observables [7] has been discussed. In this paper, unitarization is discussed. Hereby, due to the results of the preceding papers, the knowledge about a representation of the state space can be assumed. Hence we are allowed to apply this unitarization procedure to the spinor field which originally was applied to the Lee-model by Heisenberg.

Karowski considered unitarization for a relativistic field with dipole ghost regularization [8]. He modified the Heisenberg method in order to avoid explicit state representations. We shall show that the Karowski method only has formal character since so far kinematical subsidiary conditions have been ignored which prevent a successful application. Hence the explicit state construction is also unavoidable for the unitarization procedure in nonlinear spinor field theory.

We shall give a formal deduction of a correct unitarization procedure in the two fermion sector and additionally correct the dipole ghost assumptions of Heisenberg. After having obtained formal unitary, the question arises whether this formal unitarity also removes unphysical cuts or not. To answer this question, in subsequent papers model calculations will be presented to demonstrate that formal unitarization will also lead to physically meaningful results.

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1. Regularized Propagators

To achieve a relativistic invariant regularization of the nonlinear spinor field Heisenberg proposed to use a dipole regularized propagator. The introduction of such a propagator is an a priori information which has to be justified by selfconsistent calculation. But this will not be discussed here. We rather are interested in the consequences of this assumption for the state space. In theories with positive metric the connection between state space and propagator is given by the Lehmann-Kallen theorem. For indefinite state spaces this theorem has to be slightly generalized. We consider this generalization for the case of the dipole regularized spinor field.

We assume $\Psi_\alpha(x)$ to be the Hermitean spinor field operator and the set of states $\{|\varrho\rangle\}$ to be the one-particle fermion sector of the nonlinear field including ghost and dipole ghost states. Then we have the relations

$$\langle\varrho'|\varrho\rangle = g_{\varrho'\varrho} \tag{1.1}$$

and

$$\langle 0|\Psi_\alpha(x')\,\Psi_\beta(x)|0\rangle =: F^+_{\alpha\beta}(x', x) \tag{1.2}$$

where $g_{\varrho'\varrho}$ is the metrical tensor of the one-particle fermion sector and $|0\rangle$ is the groundstate of the field. If we further define

$$\varphi_\alpha(x|\varrho) := \langle 0|\Psi_\alpha(x)|\varrho\rangle \tag{1.3}$$

and assume the expansion

$$|\varrho\rangle = \int \sigma^\alpha(x|\varrho)\,\Psi_\alpha(x)|0\rangle\,\mathrm{d}^4x + \dots \tag{1.4}$$

then we obtain

$$\varphi_\beta(x'|\varrho) = \int F^+_{\beta\alpha}(x', x)\,\sigma^\alpha(x|\varrho)\,\mathrm{d}^4x + \dots . \tag{1.5}$$

The dots in (1.4) and (1.5) indicate that in general higher order contributions have to be used in order to describe the state correctly, cf. Stumpf [9]. Here we only consider the "free field" version of the propagator and omit these dots. Then the following theorem holds:

Theorem. If the sets $\{\Psi_\alpha(x)|0\rangle\}$ and $\{|\varrho\rangle\}$ are equivalent, i.e. if the linear space L_1 of the one-particle fermion sector is given by

$$L_1 = l\,h\{\Psi_\alpha(x)|0\rangle\} = l\,h\{|\varrho\rangle\},$$

then for the two-point function $F^+(x, x')$ the spectral decomposition

$$F^+_{\alpha\beta}(x, x') = \sum_{\varrho\varrho'} \varphi_\alpha(x|\varrho)^\times g^{\varrho\varrho'}\,\varphi_\beta(x'|\varrho') \tag{1.6}$$

holds.

Proof. Due to the equivalence not only (1.4) is valid but also the inversion

$$\Psi_\alpha(x)|0\rangle = \sum_\varrho b_\alpha(x|\varrho)|\varrho\rangle . \tag{1.7}$$

From this follows due to the Hermitecity of the field

$$\langle\varrho'|\Psi_\alpha(x)|0\rangle = \varphi_\alpha(x|\varrho')^\times = \sum_\varrho b_\alpha(x|\varrho)\,g_{\varrho'\varrho} \tag{1.8}$$

and

$$\begin{aligned} F^+_{\beta\alpha}(x', x) &= \sum_\varrho b_\alpha(x|\varrho)\langle 0|\Psi_\beta(x')|\varrho\rangle \\ &= \sum_\varrho \varphi_\beta(x'|\varrho)\,b_\alpha(x|\varrho) . \end{aligned} \tag{1.9}$$

By definition we have

$$\sum_{\varrho'} g^{\lambda\varrho'} g_{\varrho'\varrho} = \delta_\varrho{}^\lambda \tag{1.10}$$

and applying this to (1.8) we obtain

$$b_\alpha(x|\lambda) = \sum_{\varrho'} g^{\lambda\varrho'}\,\varphi_\alpha(x|\varrho')^\times . \tag{1.11}$$

Substitution of (1.11) into (1.9) gives (1.6), q.e.d.

The quantum number ϱ of the one-particle fermion states contains the momentum and the spin quantum numbers and additionally the specification of the particles by mass etc. Only the latter is of interest, as the momentum and spin parts are always diagonal in the model considered. Hence we have $g_{\varrho\varrho'} = \mathbf{1} \otimes \tilde g_{rr'}$ with $|\varrho\rangle = |p_\varrho, s_\varrho, r\rangle$. In the Lee-model version of Heisenberg $\tilde g_{rr'}$ is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ describing a good ghost and a dipole ghost. Assuming that this couple should accompany a physical fermion the metrical fundamental tensor has to have the structure

$$\tilde g_{rr'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{1.12}$$

Unfortunately, the spinor field propagator proposed by Heisenberg has not this simple structure but rather a more complicated one, since additional bad ghosts and parity mixtures appear [10]. In order to get a clear insight into the mechanism of unitarization and to obtain a parity invariant theory we use a modified spinor field propagator which contains only that minimum of indefiniteness being acceptable for a successful solution of the problem. We consider the general propagator

$$F(x-x') = \frac{1}{(2\pi)^4}\int \varrho_1(m'^2)\,\frac{(-\bar{p}+\mu)^2(-\bar{p}+m')}{(p^2-\mu^2)^2(p^2-m'^2)}\,e^{ip(x-x')}\,\mathrm{d}m'^2\,\mathrm{d}^4p$$
$$+\frac{1}{(2\pi^4)}\int \varrho_2(m'^2)\,\frac{(-\bar{p}-\mu)^2(-\bar{p}-m')}{(p^2-\mu^2)^2(p^2-m'^2)}\,e^{ip(x-x')}\,\mathrm{d}m'^2\,\mathrm{d}^4p\,. \quad * \tag{1.13}$$

If we only admit one parity by putting $\varrho_2 \equiv 0$ and if we assume $\varrho_1(m'^2) = \delta(m^2 - m'^2)$ then the propagator (1.13) corresponds to the propagator of a free field $\chi(x)$ with the equation

$$(-i\gamma_\mu \partial^\mu + m)(-i\gamma_\nu \partial^\nu + \mu)^2 \chi(x) = 0\,. \tag{1.14}$$

For this field we have

$$F(x-x') \equiv \langle 0|\,T\,\chi(x)\,\bar{\chi}(x')\,|0\rangle = \frac{1}{(2\pi)^4}\int \frac{(-\bar{p}+\mu)^2(-\bar{p}+m)}{(p^2-\mu^2+i\,\varepsilon)^2(p^2-m^2+i\,\varepsilon)}\,e^{ip(x-x')}\,\mathrm{d}^4p \tag{1.15}$$

with

$$(-i\gamma_\mu \partial^\mu + m)(-i\gamma_\nu \partial^\nu + \mu)^2 F(x-x') = -i\,\delta^4(x-x')\,. \tag{1.16}$$

The length dimension of $\chi(x)$ is $-\frac{1}{2}$, i.e. the local interaction is renormalizable and additionally the field is noncanonical as

$$[\chi(x)\,\bar{\chi}(x')]_{+/x_0=x_0'} = 0 \tag{1.17}$$

holds. We do not give here a detailed account of this higher order field, but rather consider the properties of the propagator. From (1.15) follows

$$F^+(x-x') \equiv \langle 0|\,\chi(x)\,\bar{\chi}(x')\,|0\rangle = \frac{1}{(2\pi)^4}\int_{C^+} \frac{(-\bar{p}+\mu)^2(-\bar{p}+m)}{(p^2-\mu^2)^2(p^2-m^2)}\,e^{ip(x-x')}\,\mathrm{d}^4p\,. \tag{1.18}$$

We decompose the fraction in (1.18) into partial fractions and obtain

$$\frac{(-\bar{p}+\mu)^2(-\bar{p}+m)}{(p^2-\mu^2)^2(p^2-m^2)} = -\frac{1}{(\mu-m)^2}\,\frac{(-\bar{p}+\mu)}{(p^2-\mu^2)}$$
$$-\frac{1}{(\mu-m)}\,\frac{(-\bar{p}+\mu)^2}{(p^2-\mu^2)^2} + \frac{1}{(\mu-m)^2}\,\frac{(-\bar{p}+m)}{(p^2-m^2)}\,. \tag{1.19}$$

Substitution of (1.19) into (1.18) and integration over p_0 gives by observing

$$(\bar{p}-\mu) = 2\mu \sum_{s=1}^{2} v_\alpha{}^s\,\bar{v}_\beta{}^s$$

the expression

$$F^+(x-x') = \frac{1}{(\mu-m)^2}\sum_s \int \frac{\mathrm{d}^3p}{\omega(m)}\,m\,\langle 0|\,\chi(x)\,|p\,s\,n\rangle\langle p\,s\,n|\,\bar{\chi}(x')\,|0\rangle\,\frac{i}{(2\pi)^3}$$
$$-\frac{1}{2(\mu-m)}\sum_s \int \frac{\mathrm{d}^3p}{\omega(\mu)}\,\mu\,\langle 0|\,\chi(x)\,|p\,s\,g\rangle\langle p\,s\,d|\,\bar{\chi}(x')\,|0\rangle\,\frac{i}{(2\pi)^3}$$
$$-\frac{1}{2(\mu-m)}\sum_s \int \frac{\mathrm{d}^3p}{\omega(\mu)}\,\mu\,\langle 0|\,\chi(x)\,|p\,s\,d\rangle\langle p\,s\,g|\,\bar{\chi}(x')\,|0\rangle\,\frac{i}{(2\pi)^3} \tag{1.20}$$

with $\omega(m) := (\mathfrak{p}^2 + m^2)^{1/2}$ and

$$\langle 0|\,\chi(x)\,|p\,s\,n\rangle := v^s(p)\,e^{ip(m)x}\,,\quad \langle 0|\,\chi(x)\,|p\,s\,g\rangle := v^s(p)\,e^{ip(\mu)x}\,, \tag{1.21}$$
$$\langle 0|\,\chi(x)\,|p\,s\,d\rangle := \left[\frac{1}{\mu-m}+\frac{1}{2\,\omega(\mu)^2}+\frac{\gamma_0}{2\,\omega(\mu)}-\frac{i\,x_0}{\omega(\mu)}\right] v^s(p)\,e^{ip(\mu)x}\,.$$

* Anm. d. Redaktion: Aus technischen Gründen konnte das Feynmann-Symbol (hier $\bar{p}$) nicht wie üblich mit dem entsprechen den Querstrich gesetst werden.

In this case the metrical tensor $g_{\varrho\varrho'} \equiv g(psr, p's'r')$ can be divided into a diagonal and a nondiagonal part, with

$$g(p\,s\,r, p'\,s'\,r') = \delta_{pp'}\,\delta_{ss'}\,\tilde{g}_{rr'} \tag{1.22}$$

where $\tilde{g}_{rr'}$ is given by (1.12) with $r \equiv n, g, d$; i.e. a physical fermion ($\equiv$ nucleon), a good ghost and a dipole ghost. The scalar products can be written

$$\tilde{g}_{rr'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \langle n|n\rangle & \langle n|g\rangle & \langle n|d\rangle \\ \langle g|n\rangle & \langle g|g\rangle & \langle g|d\rangle \\ \langle d|n\rangle & \langle d|g\rangle & \langle d|d\rangle \end{pmatrix}. \tag{1.23}$$

This means that $\langle g|g\rangle = 0 = \langle d|d\rangle$ and that the physical part of $\tilde{g}_{rr}$ can be defined by

$$(\tilde{g}_{rr})_{\mathrm{Ph}} = \begin{pmatrix} \langle n|n\rangle & \langle n|g\rangle \\ \langle g|n\rangle & \langle g|g\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.24}$$

as the good ghost states do not contribute to the norm.

2. Kinematical Conditions

As can be seen by diagonalization of (1.12), the introduction of ghost states (and dipole ghost states) leads to indefinite metric in state space. This prevents a direct probabilistic interpretation of the state space like in conventional theories with positive definite metric. To remove this unphysical feature of the theory special procedures have to be performed. As the global observables given by a complete set of commuting symmetry observables and the S-matrix are sufficient for the description of microphysical experiments, only these observables are considered. While the symmetry observables can simply be projected into the positive part of the state space, the S-matrix in general becomes nonunitary and requires a unitarization procedure in order to obtain physical meaning. As in the nonlinear spinor theory the various particle sectors have to be derived by construction, i.e. direct calculation, and are not given a priori, there does not exist a unitarization procedure which solves the problem for all sectors simultaneously. One rather has to discuss the state vectors of any sector separately. To demonstrate the method we give a discussion of elastic nucleon-nucleon scattering. The in- and outgoing states are defined in this case by the direct product of the one-fermion sector discussed in section 1, namely

$$\{|p_1\,s_1\,n\rangle, |p_1\,s_1\,g\rangle, |p_1\,s_1\,d\rangle\} \otimes \{|p_2\,s_2\,n\rangle, |p_2\,s_2\,g\rangle, |p_2\,s_2\,d\rangle\}. \tag{2.1}$$

But only the set $\{|n p_1 s_1\rangle \otimes |n p_2 s_2\rangle\}$ has a physical meaning. However, due to (1.24) also the enlarged space

$$\mathfrak{H}_{\mathrm{Ph}} := \{|n\,p_1\,s_1\rangle, |g\,p_1\,s_1\rangle\} \otimes \{|n\,p_2\,s_2, |g\,p_2\,s_2\rangle\}$$

can be considered as a physical one.

In general, the scattering states have to be calculated by means of channel equations which have been derived in preceding papers by Stumpf [11]. These states contain unphysical admixtures produced by the special form of the dynamics in consideration. That means that if one starts in the physical sector by defining the initial states to be real nucleons, the dynamical evolution causes the occurrence of ghost particles which appear in the scattering states, and this is equivalent to the loss of unitarity of the S-matrix. We discuss this situation for elastic nucleon-nucleon scattering and denote the advanced and retarded scattering states of two ingoing respectively outgoing nucleons by $|n k_1 n k_2\rangle^{(\pm)}$ where we for brevity omit the spin variables and other quantum numbers. Then the S-matrix is given by

$$S(n\,k_1\,n\,k_2\,|\,n\,k_1'\,n\,k_2') := {}^{(-)}\langle n\,k_1\,n\,k_2\,|\,n\,k_1'\,n\,k_2'\rangle^{(+)} \tag{2.2}$$

where the scalar product has to be evaluated according to Stumpf [12]. This evaluation shall not be discussed here; we only study the unitarization procedure. Because of the properties of the scalar product (2.2) can be written

$$\begin{aligned} S(n\,k_1\,n\,k_2/n\,k_1'\,n\,k_2') &= \delta(\mathfrak{k}_1 - \mathfrak{k}_1')\,\delta(\mathfrak{k}_2 - \mathfrak{k}_2') \\ &\quad + \delta(k_1 + k_2 - k_1' - k_2')\,T(s,t) \end{aligned} \tag{2.3}$$

with the usual definition of the variables s and t occurring in (2.3). In general, neither S nor T satisfy the usual unitary conditions.

Heisenberg performed a dipole ghost regularization for the Lee-model and achieved a unitary S-matrix by admixing suitable nonphysical ghost states to the scattering states [13]. However, this method requires an explicit state construction. Karowski proposed an extension of the Heisenberg method to the dipole ghost regularized nonlinear spinor field [14]. He tried to restore unitarity by a suitable admixture of nonphysical S-matrix elements to the nonunitary S-matrix and in this way to avoid the direct construction of states. This method, however, underlies a kinematical restriction. An admixture is only possible if the quantum numbers of the admixed elements coincide with those of the original S-matrix element (2.2) resp. (2.3). If this condition cannot be satisfied, an interpretation of the process in consideration is not possible. Thus, we have to consider those configurations which lead to the formulation of kinematical unitarization conditions.

It is convenient to discuss these conditions in the center of mass-system. For the original process (2.2) we have in the C-M-system

$$\mathfrak{k}_1 + \mathfrak{k}_2 = 0 = \mathfrak{k}_1' + \mathfrak{k}_2' \tag{2.4}$$

and therefore

$$\mathfrak{k}_1 = -\mathfrak{k}_2; \quad \mathfrak{k}_1' = -\mathfrak{k}_2'. \tag{2.5}$$

Together with four momentum conservation

$$k_1 + k_2 = k_1' + k_2' \tag{2.6}$$

follows then

$$|\mathfrak{k}_1| = |\mathfrak{k}_2| = |\mathfrak{k}_1'| = |\mathfrak{k}_2'| \tag{2.7}$$

and

$$k_{10} = k_{20} = k_{10}' = k_{20}' \tag{2.8}$$

with $k_0 = (\mathfrak{k}^2 + m^2)^{1/2}$. The variable s is given by

$$\begin{aligned} s &= (k_1 + k_2)^2 = (k_1' + k_2') \\ &= -4(\mathfrak{k}_1^2 + m^2) \end{aligned} \tag{2.9}$$

and the variable t by

$$\begin{aligned} t &= (k_1 - k_1')^2 = (k_2 - k_2')^2 \\ &= k_1^2 + k_1'^2 - 2 k_1 k_1' \\ &= -2m^2 + 2(\mathfrak{k}_1^2 + m^2) - 2\mathfrak{k}_1^2 \cos\Theta_{\mathfrak{k}_1\mathfrak{k}_1'}. \end{aligned} \tag{2.10}$$

From Heisenberg's and Karowski's approach follows that only good ghost configurations have to be admixed. For these admixtures the following initial and final configurations and hence S-matrix elements have to be considered in our case

a) $(n p_1\ n p_2) \to (n p_1'\ g q_2')$,
b) $(n p_1\ g q_2) \to (n p_1'\ n p_2')$,
c) $(n p_1\ n p_2) \to (g q_1'\ g q_2')$,
d) $(g q_1\ g q_2) \to (n p_1'\ n p_2')$,
e) $(n p_1\ g q_2) \to (n p_1'\ g q_2')$,
f) $(g q_1\ g q_2) \to (g q_1'\ g q_2')$,
g) $(g q_1\ g q_2) \to (n p_1'\ g q_2')$,
h) $(n p_1\ g q_2) \to (g q_1'\ g q_2')$.

For these processes the following theorem holds:

Theorem. For the elastic scattering process $k_1, k_2 \to k_1', k_2'$ with the corresponding variables s and t given by (2.9), (2.10) a selfconsistent admixture of the processes a)—h) with the same s and t value in the full range is only possible for the processes e) and f).

Proof. We consider the process a) in detail while for all other processes only results are given.

a) In the C-M-system we have

$$\mathfrak{p}_1 + \mathfrak{p}_2 = 0 = \mathfrak{p}_1' + \mathfrak{q}_2' \tag{2.11}$$

and therefore

$$\mathfrak{p}_1 = -\mathfrak{p}_2; \quad \mathfrak{p}_1' = -\mathfrak{q}_2'. \tag{2.12}$$

From this and from four momentum conservation

$$p_1 + p_2 = p_1' + q_2' \tag{2.13}$$

follows due to $m \neq \mu$

$$|\mathfrak{p}_1| = |\mathfrak{p}_2| \neq |\mathfrak{p}_1'| = |\mathfrak{q}_2'| \tag{2.14}$$

and

$$p_{10} = p_{20} \neq p_{10}' \neq q_{20}'. \tag{2.15}$$

The variable s is given by

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1' + q_2')^2 \\ &= -(p_{10} + p_{20})^2 = -(p_{10}' + q_{20}')^2 \end{aligned} \tag{2.16}$$

and the numerical equality with (2.9) requires

$$s = -(p_{10} + p_{20})^2 = -4(\mathfrak{k}_1^2 + m^2) \tag{2.17}$$

and

$$s = -(p_{10}' + q_{20}')^2 = -4(\mathfrak{k}_1^2 + m^2). \tag{2.18}$$

This can be satisfied by direct calculation with

$$\mathfrak{p}_1^2 = \mathfrak{p}_2^2 = \mathfrak{k}_1^2 \tag{2.19}$$

and

$$\mathfrak{p}_1'^2 = \mathfrak{q}_2'^2 = \frac{(4\mathfrak{k}_1^2 + 3m^2 - \mu^2)^2 - 4m^2\mu^2}{16(\mathfrak{k}_1^2 + m^2)}. \tag{2.20}$$

The variable t is given by

$$\begin{aligned} t &= (p_1 - p_1')^2 \\ &= -2m^2 + 2(\mathfrak{p}_1^2 + m^2)^{1/2}(\mathfrak{p}_1'^2 + m^2)^{1/2} \\ &\quad - 2|\mathfrak{p}_1||\mathfrak{p}_1'| \cos\Theta_{\mathfrak{p}_1\mathfrak{p}_1'} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} t &= (p_2 - q_2')^2 \\ &= -m^2 - \mu^2 + 2(\mathfrak{p}_2^2 + m^2)^{1/2}(\mathfrak{q}_2'^2 + \mu^2)^{1/2} \\ &\quad - 2|\mathfrak{p}_2||\mathfrak{q}_2'| \cos\Theta_{\mathfrak{p}_2\mathfrak{q}_2'}. \end{aligned} \tag{2.22}$$

The numerical equivalency with (2.10) requires

$$(\mathfrak{p}_1^2 + m^2)^{1/2} (\mathfrak{q}_2'^2 + m^2)^{1/2} - |\mathfrak{p}_1| \, |\mathfrak{q}_2'| \cos \Theta_{\mathfrak{p}_1 \mathfrak{p}_1'} = (\mathfrak{k}_1^2 + m^2) - |\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'} \quad (2.23)$$

and

$$-m^2 - \mu^2 + 2(\mathfrak{p}_2^2 + m^2)^{1/2} (\mathfrak{q}_2' + \mu^2)^{1/2} - |\mathfrak{p}_2| \, |\mathfrak{q}_2'| \cos \Theta_{\mathfrak{p}_2 \mathfrak{q}_2'} = -2m^2 + 2(\mathfrak{k}_1^2 + m^2) - 2|\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'} . \quad (2.24)$$

The solution of these conditions gives

$$\cos \Theta_{\mathfrak{p}_1 \mathfrak{p}_1'} = \frac{\frac{1}{4}(m^2 - \mu^2) + |\mathfrak{k}_1| \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}}{|\mathfrak{p}_1| \, |\mathfrak{q}_2'|} \quad (2.25)$$

and

$$\cos \Theta_{\mathfrak{p}_2 \mathfrak{q}_2'} = \cos \Theta_{\mathfrak{p}_1 \mathfrak{p}_1'} \quad (2.26)$$

which leads with (2.23) and (2.24) to the final expression

$$\cos \Theta_{\mathfrak{p}_1 \mathfrak{p}_1'} = \frac{[\frac{1}{4}(m^2 - \mu^2) + |\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}] \, 4(\mathfrak{k}_1^2 + m^2)^{1/2}}{|\mathfrak{k}_1| \, [(4\mathfrak{k}_1^2 + 3m^2 - \mu^2)^2 - 4m^2 \mu^2]^{1/2}} . \quad (2.27)$$

The formulas (2.19), (2.20), (2.27) have to be satisfied for the variables $(p_1 p_2, p_1' q_2')$ of the process a) in order that the s and t variables of a) have the same values as the original process. These conditions cannot be satisfied by $(p_1 p_2, p_1' q_2')$ in the full range of s and t for the process $(k_1 k_2, k_1' k_2')$. For sufficiently small $|\mathfrak{k}|$ the right hand side of (2.27) becomes >1 and therefore in this region no corresponding $(p_1 p_2, p_1' q_2')$ exist.

b) This process is equivalent to a) and leads to analogous expressions.

c) The conditions are

$$\mathfrak{p}_1^2 = \mathfrak{p}_2^2 = \mathfrak{k}_1^2 , \quad (2.28)$$

$$\mathfrak{q}_1'^2 = \mathfrak{q}_2'^2 = \mathfrak{k}_1^2 + m^2 - \mu^2 , \quad (2.29)$$

and

$$\cos \Theta_{\mathfrak{p}_1 \mathfrak{q}_1'} = \cos \Theta_{\mathfrak{p}_2 \mathfrak{q}_2'} = \frac{\frac{1}{2}(m^2 - \mu^2) + |\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}}{|\mathfrak{k}_1| \, (\mathfrak{k}_1^2 + m^2 - \mu^2)^{1/2}} . \quad (2.30)$$

For sufficiently small $|\mathfrak{k}_1|$ the right hand side of (2.30) becomes >1, hence the same conclusion as for a) is valid.

d) This process is equivalent to c) and leads to analogous expressions.

e) The conditions are

$$\mathfrak{p}_1^2 = \mathfrak{p}_1'^2 = \mathfrak{q}_2^2 = \mathfrak{q}_2'^2 = \frac{(4\mathfrak{k}_1^2 + 3m^2 - \mu^2)^2 - 4m^2 \mu^2}{16(\mathfrak{k}_1^2 + m^2)} \quad (2.31)$$

and

$$\cos \Theta_{\mathfrak{p}_1 \mathfrak{p}_1'} = \cos \Theta_{\mathfrak{q}_2 \mathfrak{q}_2'} = \frac{[\frac{1}{2}(m^2 - \mu^2) + |\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}] \, 16(\mathfrak{k}_1^2 + m^2) + (m^2 - \mu^2)^2}{(4\mathfrak{k}_1^2 + 3m^2 - \mu^2)^2 - 4m^2 \mu^2} . \quad (2.32)$$

For $\lim |\mathfrak{k}_1| = 0$ the right hand side of (2.32) becomes 1. For $\lim |\mathfrak{k}_1| \to \infty$ the right hand side of (2.32) becomes $\cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}$. So we conclude that over the full range of the original process $(k_1 k_2, k_1' k_2')$ the corresponding process e) exists.

f) The conditions are

$$\mathfrak{q}_1^2 = \mathfrak{q}_2^2 = \mathfrak{q}_1'^2 = \mathfrak{q}_2'^2 = \mathfrak{k}_1^2 + m^2 - \mu^2 \quad (2.33)$$

and

$$\cos \Theta_{\mathfrak{q}_1 \mathfrak{q}_1'} = \cos \Theta_{\mathfrak{q}_2 \mathfrak{q}_2'} = \frac{m^2 - \mu^2 + |\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}}{\mathfrak{k}_1^2 + m^2 - \mu^2} . \quad (2.34)$$

In this case a complete correspondence can be found to the original s-t-range.

g) The conditions are

$$\mathfrak{q}_1^2 = \mathfrak{q}_2^2 = \mathfrak{k}_1^2 + m^2 - \mu^2 , \quad (2.35)$$

$$\mathfrak{p}_1'^2 = \mathfrak{q}_2'^2 = \frac{(4\mathfrak{k}_1^2 - 3m^2 - \mu^2)^2 - 4m^2 \mu^2}{16(\mathfrak{k}_1^2 + m^2)} \quad (2.36)$$

$$\cos \Theta_{\mathfrak{q}_1 \mathfrak{p}_1'} = \cos \Theta_{\mathfrak{q}_2 \mathfrak{q}_2'} = \frac{[\frac{3}{4}(m^2 - \mu^2) + |\mathfrak{k}_1|^2 \cos \Theta_{\mathfrak{k}_1 \mathfrak{k}_1'}] \, 4(\mathfrak{k}_1^2 + m^2)^{1/2}}{[(4\mathfrak{k}_1^2 + 3m^2 - \mu^2)^2 - 4m^2 \mu^2]^{1/2} (\mathfrak{k}_1^2 + m^2 - \mu^2)^{1/2}} . \quad (2.37)$$

For $\lim |\mathfrak{k}_1| = 0$ and for certain ranges of m and μ (2.37) becomes > 1. E.g. if we put $m = \sqrt{2}$, $\mu = 1$ we obtain (2.37) > 1 for sufficiently small $|\mathfrak{p}_1|$. Hence no full correspondence of this process to the original process exists.

h) This process is equivalent to g) and leads to analogous expressions.

From these results follows that only the elements

$$S(n\, p_1\, g\, q_2 \,|\, n\, p_1'\, g\, q_2') =: S_{ng}^{ng}$$

and

$$S(g\, q_1\, g\, q_2 \,|\, g\, q_1'\, g\, q_2') =: S_{gg}^{gg}$$

can be admixed to S_{nn}^{nn} without any further restriction, q.e.d.

3. Unitarity Conditions

We assume to have the proper physical scattering states $|n k_1 n k_2, t\rangle_{\text{Ph}}^{(\pm)}$ which result from a unitarization procedure in the Schrödinger representation. Then, if we start in the physical sector, the scattering process must not lead into the unphysical sector. As the good ghosts are normalized to zero, the physical sector is given by

$$\mathfrak{H}_{\text{Ph}} := \{n, g\} \otimes \{n, g\},$$

while the unphysical sector follows to be

$$\mathfrak{H}_{\text{u}} := \{d\} \otimes \{d\},$$

i.e. contains the dipole ghost states. Then an advanced scattering state starting from $\mathfrak{H}_{\text{Ph}}$ must not contain outgoing bad ghosts for $t \to \infty$. Due to the orthonormality relations (1.23) no bad ghosts leave the scattering process if

$$\lim_{t\to\infty} {}^{f}\langle n\, k'\, g\, q \,|\, n\, k_1\, n\, k_2, t\rangle_{\text{Ph}}^{(+)} = 0 \tag{3.1}$$

and

$$\lim_{t\to\infty} {}^{f}\langle g\, q_1'\, g\, q_2' \,|\, n\, k_1\, n\, k_2, t\rangle_{\text{Ph}}^{(+)} = 0 \tag{3.2}$$

are satisfied where f means free two particle state. In the same way one concludes

$$\lim_{t\to-\infty} {}_{\text{Ph}}^{(-)}\langle n\, k_1'\, n\, k_2', t \,|\, n\, k\, g\, q\rangle^{f} = 0 \tag{3.3}$$

and

$$\lim_{t\to-\infty} {}_{\text{Ph}}^{(-)}\langle n\, k_1'\, n\, k_2', t \,|\, g\, q_1\, g\, q_2\rangle^{f} = 0\,. \tag{3.4}$$

Additionally, we have the normalization condition

$${}_{\text{Ph}}^{(-)}\langle n\, n \,|\, n\, n\rangle_{\text{Ph}}^{(-)} = {}_{\text{Ph}}^{(+)}\langle n, n \,|\, n\, n\rangle_{\text{Ph}}^{(+)} = 1\,. \tag{3.5}$$

Rewriting (3.1)–(3.4) in the Heisenberg representation gives

$$\begin{aligned} {}^{(-)}\langle n\, k'\, g\, q \,|\, n\, k_1\, n\, k_2\rangle_{\text{Ph}}^{(+)} &= 0\,, \\ {}^{(-)}\langle g\, q_1'\, g\, q_2' \,|\, n\, k_1\, n\, k_2\rangle_{\text{Ph}}^{(+)} &= 0\,. \end{aligned} \tag{3.6}$$

$$\begin{aligned} {}_{\text{Ph}}^{(-)}\langle n\, k_1'\, n\, k_2' \,|\, n\, k\, g\, q\rangle^{(+)} &= 0\,, \\ {}_{\text{Ph}}^{(-)}\langle n\, k_1'\, n\, k_2' \,|\, g\, q_1\, g\, q_2\rangle^{(+)} &= 0\,. \end{aligned} \tag{3.7}$$

We demonstrate now that the following theorem holds:

Theorem. A unitarization procedure satisfying the conditions (3.5), (3.6), (3.7) cannot be performed by S-matrix admixtures.

Proof. We perform the proof by explicit construction. We put

$$|n\, n\rangle_{\text{Ph}}^{(+)} = \varrho_1\, |n\, n\rangle^{(+)} + \varrho_2\, |n\, g\rangle^{(+)} + \varrho_3\, |g\, g\rangle^{(+)} \tag{3.8}$$

and

$${}_{\text{Ph}}^{(-)}\langle n\, n| = \varrho_1'\, {}^{(-)}\langle n\, n| + \varrho_2'\, {}^{(-)}\langle n\, g| + \varrho_3'\, {}^{(-)}\langle g\, g|\,. \tag{3.9}$$

By substitution of (3.8) into (3.6) and (3.9) into (3.7) we obtain

$$\varrho_2 = \varrho_1 \frac{\begin{vmatrix} S_{ng}^{nn} & S_{gg}^{nn} \\ S_{ng}^{gg} & S_{gg}^{gg} \end{vmatrix}}{\begin{vmatrix} S_{gg}^{ng} & S_{ng}^{ng} \\ S_{gg}^{gg} & S_{ng}^{gg} \end{vmatrix}} = \varrho_1 \frac{D_1(S)}{D_2(S)}\,, \tag{3.10}$$

$$\varrho_3 = \varrho_1 \frac{\begin{vmatrix} S_{ng}^{nn} & S_{gg}^{nn} \\ S_{ng}^{ng} & S_{gg}^{ng} \end{vmatrix}}{\begin{vmatrix} S_{gg}^{gg} & S_{ng}^{gg} \\ S_{gg}^{gg} & S_{ng}^{ng} \end{vmatrix}} = \varrho_1 \frac{D_1'(S)}{D_2(S)}\,, \tag{3.11}$$

and similar expressions for ϱ_2' and ϱ_3'. By using the conditions (3.5), also ϱ_1 resp. ϱ_1' can be determined. The final formula reads

$$(S_{nn}^{nn})_{\text{Ph}} = \varrho_1'^{\times} \varrho_1 \left[S_{nn}^{nn} + \frac{D_1}{D_2} S_{nn}^{ng} + \frac{D_1'}{D_2} S_{nn}^{gg} \right]. \tag{3.12}$$

This formula shows that a S-matrix admixture is not possible, since only S_{ng}^{ng} and S_{gg}^{gg} are available in the full range of s and t, but not the other elements, q.e.d.

Due to this theorem, unitarization has to be performed by the use of state representations. In this case we don't formulate the unitarity conditions for $t \to -\infty$ resp. $t \to \infty$, but for the Heisenberg states $t = 0$. This gives

$$\begin{aligned} {}^{f}\langle n\, g \,|\, n\, n\rangle_{\text{Ph}}^{(+)} &= 0\,, \\ {}^{f}\langle g\, g \,|\, n\, n\rangle_{\text{Ph}}^{(+)} &= 0\,, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} {}^{(-)}_{\text{Ph}}\langle n\,n \mid n\,g\rangle^{f} &= 0\,, \\ {}^{(-)}_{\text{Ph}}\langle n\,n \mid g\,g\rangle^{f} &= 0 \end{aligned} \tag{3.14}$$

while (3.5) remains unchanged. The sets $\{|\,nn\rangle^{(+)}_{\text{ph}}\}$ resp. $\{{}^{(-)}_{\text{ph}}\langle nn\,|\}$ are then considered to be parts of the unitarized physical Hilbert space. Imposing these conditions which lead to the direct construction of the physical Hilbert space and not only to a unitarized S-matrix, it is no more possible to reexpress all terms by S-matrix elements. The scalar products of free states with scattering states have rather to be evaluated properly, what can be done according to the methods of functional quantum theory. We only give the results.

Theorem. The unitarized part of the Hilbert space $\{|\,nn\rangle^{(+)}_{\text{ph}}\}$ for two nucleon scattering is given by

$$|\,n\,n\rangle^{(+)}_{\text{Ph}} = \varrho_1\left\{|\,n\,n\rangle^{(+)} + \frac{D_1(U)}{D_2(U)}\,|\,n\,g\rangle^{(+)} + \frac{D_1'(U)}{D_2(U)}\,|\,g\,g\rangle^{(+)}\right\} \tag{3.15}$$

where S in $D(S)$ is replaced by U with the definitions

$$U^{xy}_{zw} := {}^{f}\langle x\,y\mid z\,w\rangle^{(+)}\,. \tag{3.16}$$

Proof. Runs on the same pattern as the foregoing theorem, q.e.d.

The direct unitarization of the state space has the great advantage that no kinematical subsidiary conditions have to be imposed. Hence, this method underlies no restrictions which prevent its practical application. Either the unitary conditions (3.5), (3.13), (3.14) are satisfied by themselves, namely the process in consideration does not admit the occurrence of bad ghosts, then nothing at all has to be done, or the conditions are nontrivial and the procedure can be performed. The physical S-matrix then follows in the usual way by the formation of scalar products between the sets of states $\{|\,nn\rangle^{(+)}_{\text{ph}}\}$ and $\{{}^{(-)}_{\text{ph}}\langle nn\,|\}$. By this procedure the physical interpretation of the asymptotic states can be maintained as only states of norm zero are admixed. This is the advantage of this method.

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